THE UNIVERSITY OF ALBERTA

ORDERED GROUPS AND SOME RELATED CLASSES

by

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ABSTRACT

An ordered group is one admitting an order relation invariant under right and left multiplication. This thesis deals mostly with the group-theoretical aspect of the theory, that is with results connecting orderability properties and purely group-theoretical properties. The main results obtained in this spirit concern the classes 0^* and I^*, consisting of the groups in which every partial order can be extended respectively to a total order and to an isolated order (i.e. an order in which an element x is positive whenever it has a power x^n with $n \geq 1$ which is positive). The class of ordered groups will be denoted by 0. It is shown that polycyclic 0 - groups are 0^*, and that torsion free abelian-by-nilpotent groups are I^*, while centre-by-metabelian 0 - groups, torsion-free polycyclic groups, and abelian-by-polycyclic 0 - groups need not be I^*. Also subgroups of 0^* groups need not be I^*.

Another problem studied is that of counting in how many different ways can a group be ordered. It is shown that if the number of relatively convex subgroups of an 0 - group G is finite, then the number of orders afforded by G is also finite. If this number is greater than 2, then it is divisible by 4. Conversely, given any positive integer n, there exists an 0 - group with exactly $4n$ orders and 3 relatively convex subgroups. Some consideration is also given to classes of the type $\mathcal{X} \cap 0$, where $\mathcal{X}$ is some group theoretical class. In particular, it is shown that Engel-by-periodic 0 - groups have a central system, and that locally nilpotent-by-periodic 0 - groups are locally nilpotent. These results are

used to discuss the problem of extending an order relation of a subgroup to the whole group in the case when the group is either Engel, or locally nilpotent, or polycyclic.

Finally, some attention is given to right-ordered groups and in particular to a certain subclass containing the class of 0 - groups.

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TABLE OF CONTENTS

Page

CHAPTER I

ORDERED GROUPS

1.1 Definitions and Notations. We list here the main definitions and results that will be used in the thesis, and refer the reader to L. Fuchs' monograph [9] and to the survey articles [40] and [41] of A.A. Vinogradov for a more complete reference.

We shall adopt the terminology and notations for group-theoretical classes and closure operations introduced by P. Hall in [13] and used by Robinson in [35]. In particular, $S,P,Q,D,C,F,W,\overline{W},L,R$ will stand respectively for subgroup, extension ("poly"), quotient, direct product, cartesian product, free product, wreath product, unrestricted wreath product, locally and residually: R_o denotes the finite version of R ; $\mathcal{A},\mathcal{A}^2,\mathcal{N}_c,\mathcal{E},\mathcal{F},\mathcal{P},\mathcal{PG}_1,\mathcal{F}^{-s}$ denote respectively the classes of groups which are abelian, metabelian, nilpotent of class c , Engel, finite, periodic, polycyclic and torsion-free; if $\mathcal{X}$ and $\mathcal{Y}$ are any two group-theoretical classes, $\mathcal{XY}$ is the class of groups which are extensions of an $\mathcal{X}$ - group by a $\mathcal{Y}$ - group. See [35] for the definitions of Z - group, ZD - group and SN - group.

We shall also assume that the reader is familiar with the following terms which are defined in [9]: partial order, linear or full or total order, positive and negative cone, isolated partial order, Archimedean order, extension of a partial order, partially ordered group, ordered group or 0 - group, 0^* - group, 0 - homomorphism, 0 - isomorphism, convex

subgroup, jump, $|a|$, $a \leq b$, $a << b$, $a \sim b$, where a and b are elements of an 0 - group, $S(x_1,\ldots,x_r)$, $S'(x_1,\ldots,x_r)$, where $x_1,\ldots,x_r$ are elements of a group.

Definition 1.1.1. Let H be a normal subgroup of a group G . An order of H is a G - order if it is a partial order of G .

Definition 1.1.2. A subgroup H of a group G is called G - fully-orderable if every maximal partial order of G is linear when restricted to H .

Definition 1.1.3. A subgroup H of an 0 - group G is called relatively convex if it is convex under some order of G , absolutely convex if it is convex under every order of G .

Definition 1.1.4. A subgroup H of a group G is strongly isolated if for all $g,x_1,\ldots,x_r$ in G , $g^{x_1} \ldots g^{x_r} \in H$ implies that $g^{x_1},\ldots,g^{x_r}$ are all in H .

Note that every relatively convex subgroup is strongly isolated, for if $g \geq e$ then $g^{x_1} \ldots g^{x_r} \geq g^{x_i} \geq e$ for $i = 1,\ldots,r$; if $g \leq e$ then $g^{x_1} \ldots g^{x_r} \leq g^{x_i} \leq e$ for $i = 1,\ldots,r$.

Definition 1.1.5. A subgroup H of a group G is called infrainvariant if for any $g \in G$, $H^g \geq H$ or $H^g \leq H$. A group G is said to have an infrainvariant system $\sum$, if $\sum$ is a normal system (as defined by Kurosh

in [24] p. 171) such that every $C \in \sum$ is an infrainvariant subgroup of G .

1.1.6 (Kokorin [21]). Let G be an 0 - group and K a normal, relatively convex, G - fully-orderable subgroup of G . If H is an infrainvariant strongly isolated subgroup of G and $H \leq K$, then H is relatively convex in G .

A proof of 1.1.6 is supplied in [38] p. 20.

1.1.7 (Kokorin [22]). An intersection of relatively convex subgroups is relatively convex if and only if it is infrainvariant.

1.2 Basic Characterizations of 0 - Groups. The notion of convex subgroup plays an indispensable role in any study of the structure of 0 - groups. After Hölder's classical theorem on Archimedean 0 - groups, came the results of Iwasawa, Rieger, Podderyugin, Mal'cev and Kokorin characterizing 0 - groups in terms of a complete infrainvariant system satisfying certain conditions. Other characterizations were given by Łoś Ohnishi, Sevrin and Tully. We list here the ones that we have found more useful. See [9] for proofs.

1.2.1 (Hölder). An 0 - group G is Archimedean if and only if it is 0 - isomorphic to a subgroup of the additive group of the real numbers under the natural ordering.

1.2.2 (Rieger, Podderyugin). A normal subgroup H of a group G admits a G - order if and only if it contains a system $\sum$ of subgroups satisfying

the following conditions:

(1) $\sum$ is a chain containing $\{e\}$ and H ; together with the subgroups C_λ $(\lambda\in\Lambda)$ their intersection $\cap C_\lambda$ and their union $\cup C_\lambda$ also belong to $\sum$;

(2) if $C \in \sum$ and $g \in G$, then $g^{-1}C\, g \in \sum$;

(3) if $C \prec D$ is a jump in $\sum$, then $C \Delta D$ and D/C is isomorphic to a subgroup of the real numbers;

(4) if $C \prec D$ is a jump in $\sum$, then $[N(C),N(C),D] \leq C$;

(5) if $C \in \sum$ and $S(a)$ intersects C for some $a \in H$, then some conjugate of a lies in C .

1.2.3 (Łoś, Ohnishi). A normal subgroup H of a group G admits a G - order if and only if for every finite set of elements $a_1,\ldots,a_r \in H \setminus \{e\}$, there are signs $\varepsilon_i = \pm 1$ such that $e \notin S(a_1^{\varepsilon_1},\ldots,a_r^{\varepsilon_r})$.

1.3 <u>Closure Properties of the Class of 0 - Groups</u>.

1.3.1 $0 = \{S,C,R,W,L,F\}\, 0$.

The S - closure and the C - closure are easily verified and together imply the R - closure. The W - closure was proved in [11] and [28], and the L - closure in [29]. The F - closure is due to A.A. Vinogradov [39].

1.3.2 The class 0 is not closed under any of the closure operations $Q,P,\overline{W}$.

The quotient of an 0 - group G with respect to a normal subgroup H is ordered if and only if H is relatively convex. The semi-direct product of two infinite cyclic groups may fail to be ordered, e.g. $G = \langle a,b \ ; \ b^a=b^{-1} \rangle$. The unrestricted wreath product of any two nontrivial 0 - groups can never be an 0 - group because it always contains some element other than e that is conjugate to its inverse, [30].

1.3.3 (Kokorin, Kopytov). A group G is an 0 - group if and only if both its centre Z and its central factor G/Z are 0 - groups.

A proof of 1.3.3 is supplied in [10].

1.4 <u>Order-antomorphisms of Subgroups of the Additive Group of the Real Numbers</u>. In this section we give some consequences of the following result.

1.4.1 (Hion [14]). If A and B are subgroups of the additive group of the real numbers under the natural ordering, and ϕ an 0 - homomorphism from A to B , then $\phi(a) = ar$ for some positive number r and for all $a \in A$.

1.4.2 Let $C \prec D$ be a jump in an 0 - group G . If for some $x \in G$ and $d \in D\backslash C$, $[d,x] \in C$, then $[D,x] \subseteq C$.

<u>Proof</u>. $[d,x] \in C$ implies $d \sim d^x$, hence if $a \sim d$, $a^x \sim d^x \sim d$ and if $a \ll d$, $a^x \ll d^x \sim d$ and therefore $d \in N_G(D)$. By 1.4.1, the

action of x on D/C is that of multiplication by a positive real number on the additive subgroup D/C of the real numbers. Thus if x fixes one non-zero element of D/C, then it must fix all of them.

1.4.3 Let $C \prec D$ be a convex jump in an 0 - group G and $x \in N_G(D)$. If D/C is finitely generated, then the action of x on D/C is that of multiplication by a positive algebraic integer. Thus there is an integer monic polynomial $p(t) = \sum_{i=1}^{n} \alpha_i t^i$, irreducible over the rational field, such that $d^{p(x)}$ (i.e. $\prod_{i=1}^{n} (d^{\alpha_i})^{x^i}$) lies in C for all $d \in D$. Moreover at least one root of $p(t)$ is a positive real number and $\alpha_0 = \pm 1$.

1.5 <u>Examples of 0 - Groups</u>. Using the general criteria exposed in section 1.2 one can see that groups with a central series with torsion-free factors are 0 - groups. In particular free groups and torsion-free nilpotent groups are 0 - groups. By applying suitable closure operations one can build larger classes of 0 - groups. Other means of finding 0 - groups are provided by the following theorems.

1.5.1 (Baumslag [2]). Let F be a free group, R a normal subgroup of F, S a fully invariant subgroup of R. If F/R and R/S are ordered, so is F/S.

Thus, for example, free polynilpotent groups are ordered. The above includes as a special case Snirnov's result that if F/R is ordered then F/R' is also ordered, (see 6.2.4).

1.5.2 The free centre-by metabelian group of rank 2 is an 0 - group.

Proof. This group has been proved to be torsion-free [34], therefore its centre is ordered. Its central quotient, being free metabelian, is also ordered, hence by 1.3.3 the group itself is ordered.

1.6 R^* - Groups. A group G is an R - group if for any $a,b \in G$, $a^n = b^n$, $n \neq 0$, implies $a = b$. G is an R^* - group if for any $a,b,x_1,\ldots,x_n \in G$, $a^{x_1} \ldots a^{x_n} = b^{x_1} \ldots b^{x_n}$ implies $a = b$. This is equivalent to saying that $e \notin S(g)$ for all $g \in G \setminus \{e\}$, or that the subgroup $\langle e \rangle$ is strongly isolated.

Clearly 0 - groups are R^*. It was a long standing open question whether the classes 0 and R^* coincided. The question has been recently answered in the negative by V.V. Bludov [4].

Finally we quote here a result that will be used in the next chapters (see [24], p. 243).

1.6.1 If G is an R - group then for any $a,b \in G$ and integers $m,n \neq 0$, $[a^m,b^n] = e$ implies $[a,b] = e$.

CHAPTER II

SOME PARTICULAR CLASSES OF ORDERED GROUPS

2.1 <u>0 - Groups with Normal Relatively Convex Subgroups</u>. It is evident from 1.2.2 that the structure of an 0 - group G depends on the structure of the system of its convex subgroups. In particular if the convex subgroups of G are all normal, then G' has a central system with torsion-free factors. A sufficient condition that assures the normality of infrainvariant, and hence of relatively convex subgroups is the following.

2.1.1 (A.H. Rhemtulla). If G is a group such that for any a,x in G, $\langle a^{\langle x\rangle}\rangle = \langle a^{g_1},\dots,a^{g_r}\rangle$ for some $g_1,\dots,g_r$ in G, then any infrainvariant subgroup of G is normal in G.

<u>Proof</u>. Suppose that $H < H^x$ for some $H \leq G$ and $x \in G$. Then for some $a \in H$, $a^x \in H^x \backslash H$. By hypothesis $\langle a^{\langle x\rangle}\rangle = \langle a^{g_1},\dots,a^{g_r}\rangle \leq \bigcup_{i=1}^{\infty} H^{x^i}$. Thus $a^{g_1},\dots,a^{g_r} \in H^{x^k}$ for some k and $a^{x^{k+1}} \in H^{x^k}$, so that $a^x \in H$.

Note that a group satisfying the maximal condition locally satisfies the hypothesis of 2.1.1. The following remarks are quoted from [18].

2.1.2 If G is an 0 - group satisfying the maximal condition locally, then G' is a Z - group. If G satisfies the maximal condition, then G' is a ZD - group.

2.1.3 If G is a (locally) polycyclic 0 - group, then it is (locally) nilpotent-by-abelian: $LP\,G_1 \cap 0 \subseteq L(NA)$.

In the next section we shall make use of the following

Lemma 2.1.4. Let G be an 0 - group and A a subgroup of G such that for any $g \in G$, $g^n \in A$ for some integer $n \neq 0$. If every convex subgroup of A is normal in A , then every convex subgroup of G is normal in G .

Proof. Let C be convex in G , then $C \cap A$ is convex in A and by hypothesis normal in A . Take any $g \in G$, then $g^n \in A$ for some $n \neq 0$. Also for any $c \in C$, $c^m \in C \cap A$ for some $m \neq 0$, hence $(c^m)^{g^n} \in C \cap A$. Since C is convex in G , it is isolated and therefore $c^{g^n} \in C$. This implies that $c^g \in C$ and thus we may conclude that $C^g = C$ for any $g \in G$.

2.2 The Classes of Engel-by-Periodic and of Locally Nilpotent-by-Periodic 0 - Groups.

Theorem 2.2.1. Let G be an 0 - group, A an Engel subgroup of G such that for any $g \in G$, $g^n \in A$ for some integer $n \neq 0$. Then G is a Z - group.

Proof. Consider first the case when $G \in E \cap 0$. It is easy to see by induction that for any $a,x \in G$, $\langle a,a^x,\ldots,a^{x^n}\rangle = \langle a,[a,x],\ldots,[a,_n x]\rangle$. Therefore by 2.1.1 all the convex subgroups of G are normal. Let $C \prec D$

be a convex jump, and let $y \in G$. By 1.4.2, $[D,y] \leq C$ if and only if $[x,y] \in C$ for at least one $x \in D\backslash C$. Choose any $x \in D\backslash C$. If $[x,y] \notin C$, then since G is Engel there is a positive integer n such that $[x,_n y] \notin C$ but $[x,_{n+1} y] \in C$. Thus $[D,y] \leq C$ and hence $[D,G] \leq C$, and the system of convex subgroups is a central system for G.

Now consider the general case. By 2.1.4 it is still true that all the convex subgroups of G are normal. Let $C \prec D$ be a convex jump in G, x any element in D, y any element in G. Then for some positive integers m and n, $x^m \in D \cap A$ and $y^n \in A$. Since $C \cap A \prec D \cap A$ is a convex jump in A, $[x^m, y^n] \in C \cap A \leq C$. Since G/C is an 0 - group, by 1.6.1 $[x,y] \in C$ and hence $[D,G] \leq C$.

<u>Corollary 2.2.2</u>. If $G \in \mathcal{EP} \cap 0$ then G is a Z - group.

<u>Theorem 2.2.3</u>. Let G be an 0 - group, H a subgroup of G, $H \in \mathcal{N}_c$. If G has a system of generators S such that for any $g \in S$, $g^n \in H$ for some integer $n \neq 0$, then G is also nilpotent of class c.

<u>Proof</u>. Let us show by induction that for all integers i, $Z_i(H) = H \cap Z_i(G)$, where Z_i indicates the ith centre. It is obvious that $Z_i(H) \geq H \cap Z_i(G)$. Let $h \in Z_i(H)$, in order to show that $h \in Z_i(G)$ it is enough to show that $[h,g] \in Z_{i-1}(G)$ for all $g \in S$. Since $g^n \in H$, $[h,g^n] \in Z_{i-1}(H) \leq Z_{i-1}(G)$ by induction hypothesis. Since $G/Z_{i-1}(G) \in 0$, $[h,g] \in Z_{i-1}(G)$ by 1.6.1. In particular $H = Z_c(H) \leq Z_c(G)$, thus for all $g \in S$, $g^n \in Z_c(G)$ and since $G/Z_c(G) \in \mathcal{F}^{-s}$, $S \subseteq Z_c(G)$ and $G = Z_c(G)$.

Corollary 2.2.4. If $G \in N_c P \cap 0$, then $G \in N_c$.

Corollary 2.2.5. If $G \in L(NP) \cap 0$, then $G \in LN$.

Since torsion-free supersolvable groups are nilpotent-by-finite, 2.2.5 generalizes the result of [18] that any ordered locally supersolvable group is locally nilpotent.

Theorem 2.2.6. Let G be an 0 - group and H a locally nilpotent subgroup of G. If G has a system of generators S such that for any $g \in S$, $g^n \in H$ for some integer $n \neq 0$, then G is locally nilpotent.

Proof. Let K be a maximal locally nilpotent subgroup of G containing H. If $K \neq G$ then there is a $g \in S \backslash K$, and $g^n \in K$ for some positive integer n. Consider $A = \langle K,g \rangle$. We will reach a contradiction by showing that A is locally nilpotent. Let $x_i \in A$, $i = 1,\dots,r$. Then there exist $k_j \in K$, $j = 1,\dots,s$, such that $\langle x_i\ ,\ i=1,\dots,r \rangle \leq \langle g,k_j\ ,\ j=1,\dots,s \rangle = M$. Let $N = \langle g^n,k_j\ ,\ j = 1,\dots,s \rangle$. Since K is locally nilpotent and N is a finitely generated subgroup of K, N is nilpotent. Applying 2.2.1 to the group M with subgroup N, we conclude that M itself is nilpotent and therefore A is locally nilpotent.

Corollary 2.2.7. If $G \in (LN)P \cap 0$, then $G \in LN$.

Remark. All the results of this section remain valid if we substitute the class of 0 - groups with the class of R - groups.

CHAPTER III

EXTENDING ORDER RELATIONS FROM A SUBGROUP TO THE WHOLE GROUP

3.1 Introduction. In [31] B.H. Neumann and J.A.H. Shepperd proved the following.

3.1.1 If G is a torsion-free group, H a normal subgroup of G such that G/H is locally finite, and P is a G - order of H, then P can be extended to a full order of G.

It is not known whether the condition $G/H \in L\mathcal{F}$ can be replaced by $G/H \in \mathcal{P}$. It certainly can if $H \Delta G$ and $G/H \in \mathcal{P}$ imply $(G' \cap H)/[G,H] \in \mathcal{P}$.

3.2 Engel Groups. We use 3.1.1 to prove the following

Theorem 3.2.1. If G is a torsion-free Engel group, $H \Delta G$ and G/H is locally finite, then any full order of H can be extended to a full order of G.

Proof. Let P be a full order on H. It is sufficient to prove that P is stable under the inner antomorphisms of G. Let $h \in P$, $x \in G$, $n > 0$, $x^n \in H$, $h^x \in P^{-1}$. Define $h_i = [h,_i x]$, since $G \in \mathcal{E}$, $h_m = e$ for some m. Let r be the smallest integer such that $h_r, h_r^x, \ldots, h_r^{x^{n-1}}$ are either all positive or all negative, and let i be the smallest integer such that $h_{r-1}^{x^{i-1}}$ and $h_{r-1}^{x^i}$ are one positive and one negative. Say

$h_{r-1}^{x^{i-1}} \in P$. Then $h_r^{x^{i-1}} = (h_{r-1}^{-1} h_{r-1}^{x})^{x^{i-1}} = (h_{r-1}^{-1})^{x^{i-1}} h_{r-1}^{x^i} \in P^{-1}$, and consequently $h_r, h_r^x, \ldots, h_r^{x^{n-1}} \in P^{-1}$.

$$
\begin{array}{lll}
h_r \in P^{-1} & \text{implies} & h_{r-1}^{x} \leq h_{r-1} \quad , \\
h_r^{x} \in P^{-1} & \text{implies} & h_{r-1}^{x^2} \leq h_{r-1}^{x} \leq h_{r-1} \quad , \\
\vdots & & \\
h_r^{x^{n-1}} \in P^{-1} & \text{implies} & h_{r-1}^{x^n} \leq \ldots \leq h_{r-1}^{x^i} \leq \ldots \leq h_{r-1} \quad .
\end{array}
$$

Since h_{r-1} is positive, $h_{r-1}^{x^n}$ is positive, while $h_{r-1}^{x^i}$ is negative, hence $h_{r-1} = e$, contradicting the choice of r .

Note that in order to prove that all the full orders of H are G - invariant, it is sufficient to assume that $G/H \in \mathcal{P}$, rather than $G/H \in L\mathcal{F}$.

3.3 Locally Nilpotent Groups. As a corollary of 3.2.1 we have

3.3.1 If $G \in L\mathcal{N} \cap \mathcal{F}^{-s}$, $H \Delta G$, $G/H \in \mathcal{P}$, then any full order of H can be extended to a full order of G .

Proof. Just note that $L\mathcal{N} \cap \mathcal{P} \subseteq L\mathcal{F}$ and $L\mathcal{N} \subseteq \mathcal{E}$.

Actually in the case of $L\mathcal{N}$ - groups we need not even assume that H is normal.

Theorem 3.3.2. Let $G \in L\mathcal{N} \cap \mathcal{F}^{-s}$, if H is a subgroup of G such that every $g \in G$ has some power $g^n \in H$, $n \neq 0$, then every full order of H

can be extended to a full order of G .

Proof. First we prove the result in the case when $G \in F^{-s} \cap N \cap G$, using induction on the polycyclic length of G . If $G = \langle e \rangle$ or even if G is infinite cyclic, the result is trivial. Let P be a given full order of H and C the smallest non-trivial convex subgroup. Then $C \leq Z(H)$. For any $c \in C$ and $g \in G$, $[c,g^n] = e$ for some $n \neq 0$; since $G \in 0$ this implies $[c,g] = e$, thus $C \leq Z(G)$, $C \Delta G$. Let $\overline{C}$ be the isolator of C . Since $G \in N$, $\overline{C}$ consists of the roots of the elements of C , hence $\overline{C} \cap H = C$ and $H\overline{C}/\overline{C} \cong H/C$. Let P_1 be the isomorphic image in $H\overline{C}/\overline{C}$ of the order induced by P on H/C . By induction P_1 can be extended to a full order P_2 of $G/\overline{C}$. Let P_3 be the extension to $\overline{C}$ of the order induced by P on C . Since $\overline{C} \leq Z(G)$, P_3 is G - invariant. P_2 and P_3 define a full order on G which is an extension of P .

Now let $G \in LN \cap F^{-s}$. Again let P be a full order of H . If P is extendable to G at all, then the only possible extension is $Q = \{g \mid g^n \in P\ ,\ n>0\}$. Clearly $Q \cap Q^{-1} = \{e\}$ and $Q \cup Q^{-1} = G$. We must show that for any $x \in G$, $Q^x \subseteq Q$ and that $QQ \subseteq Q$. Let $g_1, g_2 \in Q$ and consider the group $A = \langle x, g_1, g_2 \rangle$. $A \in F^{-s} \cap N \cap G$ and every element of A has some power in the subgroup $A \cap H$. By what we already proved, the order induced by P on $A \cap H$ is extendable to A . Since such an extension is unique, we conclude that $Q \cap A$ is an order on A , in particular $g_1^x \in Q$ and $g_1 g_2 \in Q$.

Corollary 3.3.3. If $G \in \mathcal{O}$, $H \Delta G$, $G/H \in \mathcal{P}$, $H \in L\mathcal{N}$, then any full order of H can be extended to a full order of G .

Proof. By 2.2.7 $G \in L\mathcal{N}$. Apply 3.3.1.

Corollary 3.3.4. If $G \in L\mathcal{N} \cap \mathcal{O}$, then G can be embedded in a radicable $L\mathcal{N} \cap \mathcal{O}$ - group R .

Remark. While it is known that any group can be embedded in a radicable group, (B.H. Neumann [27]), the analogous question is not yet settled for $\mathcal{O}$ - groups.

Proof. Any $L\mathcal{N}$ - group G can be embedded in a radicable $L\mathcal{N}$ - group R (Mal'cev [25]). We can assume that R consists exactly of all the roots of the elements of G . Thus if $G \in \mathcal{O}$ every order on G can be extended to an order on R by 3.3.2.

3.4 Polycyclic Groups. One could conceivably try to extend result 3.2.1 to some other class of groups. The following example shows that polycyclic metabelian ordered groups already fail to have such a property.

3.4.1 Let A be the subgroup of the additive group of the reals generated by $1, \alpha, \alpha^2, \alpha^3$, where $\alpha = +\sqrt{\frac{1+\sqrt{5}}{2}}$ is the positive real root of $x^4 - x^2 - 1 = 0$. Let G be the split extension of A by an infinite cyclic group $\langle t \rangle$, subject to $\beta^t = \beta\alpha$ for all $\beta \in A$. Since t induces an order-preserving automorphism of A , $G \in \mathcal{O} \cap PG_1 \cap A^2$. Let $B = \langle 1, \alpha^2 \rangle$ and $H = \langle A, t^2 \rangle$. H is a normal subgroup of G of index 2

and it has a full order in which the convex subgroups are $\langle e \rangle$, B , A , H . If this order could be extended to G , then B would be the intersection of H with a convex subgroup of G . Since G is polycyclic, all its convex subgroups are normal, and B would also be normal, contradicting the fact that $B^t = \langle \alpha, \alpha^3 \rangle \neq B$.

CHAPTER IV

ON THE NUMBER OF WAYS OF ORDERING A GROUP WHICH HAS FINITELY MANY RELATIVELY CONVEX SUBGROUPS

4.1 Introduction. It is natural to ask how many different order relations does an ordered group afford. Since to every order P there corresponds the opposite order P^{-1}, if the number of orders is finite it must be an even number. In [9] Fuchs reports B.H. Neumann's conjecture that such a number must be a power of 2 both in the finite and in the infinite case. Both parts of the conjecture have been proved false: Kargapolov, Kokorin and Kopytov have found examples of groups with exactly $2^{n+1} n!$ orders, for any integer n [20], and Buttsworth has found a family of groups with a countable infinity of orders [5]. In this chapter we settle the question of which kind of numbers can occur in the case of groups with finitely many relatively convex subgroups.

4.2 The Result.

Theorem 4.2.1. If the set of the relatively convex subgroups of an O - group G is finite, then the number of ways of ordering G is finite. This number, if greater than 2, is divisible by 4. Moreover, for any integer n there exists a group that admits exactly $4n$ orders and has 3 relatively convex subgroups.

Proof. Since G has finitely many relatively convex subgroups, they are all normal and every jump is centralized by G', so that $G \in \mathcal{NA}$ (see section

2.1). Let F be the Fitting subgroup of G, then $G/F \in \mathcal{F}^{-S} \cap \mathcal{A}$ and F is either equal to G or to the isolator of G', in which case the rank of G/F is 1. Thus F is absolutely convex and every relatively convex subgroup of G minimal with respect to some given order is contained in $Z(F)$. To see this note that in a nilpotent group the centre intersects non-trivially every normal subgroup and use 1.4.2. We want to investigate more closely the structure of a convex subgroup C which is minimal with respect to some given order P. Choose any $x \in C$ and define $D_x = \langle x\rangle^G$ and $\overline{D}_x$ equal to the set $\{c \in C \mid S(c) \cap D_x \neq \phi\}$. Let us show that C is equal to $\overline{D}_x$ (which therefore is a subgroup) and is minimal in the set of all relatively convex subgroups of G. Using the facts that $C \leq Z(F)$ and $G/F \in \mathcal{A}$, we see that $\overline{D}_x$ is a strongly isolated normal subgroup of G. If $e < C < C_1 < \cdots < C_n = G$ is the chain of convex subgroups determined by P, we can apply 1.2.2 to the chain $e < \overline{D}_x < C < C_1 < \cdots < G$ and conclude that $\overline{D}_x$ is relatively convex. Consequently every minimal relatively convex subgroup of G contained in C is of the form $\overline{D}_x$. Conversely, every $\overline{D}_x$ is minimal. Suppose $\overline{D}_x \geq \overline{D}_a$, where $\overline{D}_a$ is minimal.

Since $a \in \overline{D}_x$, $a^{\sum_{i=1}^{r} p_i g^{\alpha_i}} = x^{\sum_{j=1}^{s} n_g g^{\beta_j}}$ for some $p_i \in Z^+$, $n_j, \alpha_i, \beta_j \in Z$ $g \in G$. Let $b = a^{\sum_{j=1}^{s} n_j g^{\beta_j}}$. Note that $b \neq e$, because C is archimedeanly G-orderable, hence if $a^{\sum_{j=1}^{r} n_j g^{\beta_j}} = e$, $a^{\sum_{i=1}^{r} p_i g^{\alpha_i}} = x^{\sum_{j=1}^{s} n_j g^{\beta_j}} = e$ also, contradicting $G \in R^*$. By the minimality of $\overline{D}_a$, $\overline{D}_a = \overline{D}_b$, hence $a \in \overline{D}_b$ and we can write $a^{\sum_{h=1}^{t} q_h g^{\gamma_h}} = a^{\sum_{j=1}^{s} n_j g^{\beta_j} \sum_{k=1}^{n} m_k g^{\delta_k}}$ for some $q_h \in Z^+$,

$m_k, \gamma_h, \delta_k \in Z$, if necessary replacing g by one of its roots modulo F . Since a and x belong to C , which is archimedeanly G - orderable, we have

$$x^{\sum_{h=1}^{t} q_h g^{\gamma_h}} = x^{\sum_{j=1}^{s} n_j g^{\beta_j} \sum_{k=1}^{n} m_k g^{\delta_k}} = a^{\sum_{i=1}^{r} p_i g^{\alpha_i} \sum_{k=1}^{s} m_k g^{\delta_k}} \in D_a .$$

Hence $x \in \overline{D}_a$ and $\overline{D}_x = \overline{D}_a$. C will contain finitely many subgroups of type $\overline{D}_x$, say $\overline{D}_1,\ldots,\overline{D}_n$ and these are all minimal relatively convex subgroups of G and therefore pairwise disjoint because the intersection of two normal relatively convex subgroups is relatively convex (see 1.1.7). Let us show that $n = 1$ and $C = \overline{D}_1$. Clearly $C = \bigcup_{i=1}^{n} \overline{D}_i$ because every $x \in C$ belongs to $\overline{D}_x$. If $n > 1$ choose $y \in C\backslash\overline{D}_1$ and consider the coset $y\,\overline{D}_1$. Since $y\,\overline{D}_1$ is infinite there exists $\overline{D}_i$ $(i\neq 1)$ containing two different elements $ya, yb \in y\,\overline{D}_1$. It follows that $(yb)^{-1}ya = b^{-1}a \in \overline{D}_i \cap \overline{D}_1 = \{e\}$, contradicting $ya \neq yb$. This ends the proof of our assertions about C . Let $C_1, C_2, \ldots, C_r$ be the minimal relatively convex subgroups of G , n the number of orders of G , n_i the number of orders of G in which C_i is convex, h_i the number of orders of G/C_i , k_i the number of G - orders of C_i . Then $n = \sum_{i=1}^{r} n_i = \sum_{i=1}^{r} h_i k_i$. By induction on the number of relatively convex subgroups of G , we can assume $h_i < \infty$, so all we have to show is that $k_i < \infty$. Let C be any of $C_1,\ldots,C_r$. We know that $C = \overline{D}_x$ for some $x \in C$. The number of G - orders of C is equal to the number of G - orders of D_x . To prove this, note that the G - orders of C and of D_x are precisely their CG^* - orders, where CG^* is the split extension of C by $G^* \cong G/C_G(C)$. As a group of order-automorphisms of an archimedeanly ordered group, $G^* \in F^{-s} \cap A$. Let Q be a CG^* - order of D_x , if it can be extended at all to a CG^* - order of $\overline{D}_x$, then the only

possible extension is $\overline{Q} = \{c \in C \mid S(c) \cap Q \neq \phi\}$. That $\overline{Q}$ is indeed a CG^* - order on $\overline{D}_x$ follows from the fact that CG^* is a metabelian 0 - group and therefore an 0^* - group (see 5.3.1). Let us now count the number of G - orders of D_x . If $G^* = \{e\}$, then C must be abelian of rank 1 and have exactly 2 orders. If $G^* \neq \{e\}$, choose $g \in G^*\backslash\{e\}$. Then every $y \in G^*$ is equal to g^q for some rational number q . Let $T = \{q \mid \exists\, y \in G^*\ y = g^q\}$ and define, for every real number r , A_r to be the isolator of the group $\langle x^{g^q} \mid q \in T\ ,\ q < r\rangle$ and B_r the isolator of the group $\langle x^{g^q} \mid q \in T\ ,\ q \leq r\rangle$. By 1.2.2 $\{\langle e\rangle, A_r, B_r, D_k\}$ are convex subgroups in some G - order of D_x . Since every G - order of D_x is extendable to a G - order of C and these are all archimedean, it must be for all r $A_r = B_r = D_x$. In particular, $x \in A_o$ i.e. $x^k = x^{\sum_{j=1}^{\ell} n_j g^{q_j}}$ for some integers k and n_j and rationals $q_1 < \cdots < q_\ell < 0$. Replacing if necessary g with $g^{1/m}$, where m is the l.c.m. of the denominators of the q_j's , we can assume that $q_1,\ldots,q_\ell$ are also integers. Conjugating by g^{-q_1} we obtain $x^{f(g)} = e$ where $f(g)$ is a polynomial of degree $1-q_1$ with integer coefficients. Without loss of generality we can also assume it to be irreducible over Q . The number of G - orders of D_x is equal to twice the number of non-equivalent monomorphisms ϕ of D_x into the additive group of real numbers which represent conjugation by elements of G as multiplication by positive reals. Two such monomorphisms are equivalent if they differ by an order-automorphism of $(R,+)$, thus we can assume $\phi(x) = 1$, and every monomorphism will be determined by the image of x^g . In fact every $d \in D_x$ can be written as

$d = x^{\sum_{i=1}^{s} m_i g^{t_i}}$ for some $m_i \in Z$, $t_i \in T \subseteq Q$, and therefore, if

$\phi(x^g) = \alpha \in R^+$, $\phi(d) = \sum_{i=1}^{s} m_i \alpha^{t_i}$, where the roots of α involved, if any, are the positive real ones. From $\phi(x^{f(g)}) = \phi(e) = 0$, it follows that $f(\alpha) = 0$, i.e. α is a positive real root of the polynomial f , so that there are only finitely many choices for α and consequently finitely many G - orders on D_x . This ends the proof that the number of orders of G is finite. From the formula $n = \sum_{i=1}^{r} h_i k_i$ we see that if G has any proper non-trivial relatively convex subgroup n must be divisible by 4 , otherwise G is abelian of rank 1 and has exactly 2 orders.

Let us now give examples of groups with 3 relatively convex subgroups admitting exactly $4n$ orders for every positive integer n . Suppose that $f(x)$ is a monic polynomial of degree $m > 1$, with integer coefficients, irreducible over Q , and with constant term equal to ± 1 . Suppose further that exactly $n \geq 1$ of the roots of $f(x)$ are positive real numbers. Let

$$G = \langle a_1,\ldots,a_m,g; [a_i,a_j] = e \quad \text{for } i,j = 1,\ldots,m \quad ,$$

$$a_1^{g^i} = a_{i+1} \quad \text{for } i = 1,\ldots,m-1, a_1^{f(g)} = e\rangle \quad .$$

G is a polycyclic metabelian group, its Fitting subgroup is the torsion free abelian group $A = \langle a_1,\ldots,a_m\rangle$. G is an O - group, its only proper non-trivial relatively convex subgroup is A . G/A is infinite cyclic and has exactly 2 orders, while the number of G - orders of A is twice the number of positive real roots of $f(x)$, i.e. $2n$. Thus G has $4n$ orders,

as required. We still have to prove the existence of $f(x)$. To do this we use the following definition and results from L. Bernstein, "The Jacobi-Perron Algorithm", p. 119.

"Definition X. A polynomial $F(x)$ of degree ≥ 2 is called a third degree P - polynomial if it has the form $F(x) = (x-D)(x-D_1)\ldots(x-D_{n-1})-d$; where

$$D, D_i \,,\; d \in Z \,,$$

$$D \equiv D_i \pmod{d} \quad i = 1,\ldots,n-1 \,,$$

$$d \geq 1 \quad D = D_o > D_1 > \ldots > D_{n-1} \,,$$

$$D_1-D_2 \geq 2 \quad \text{or} \quad D_o-D_1 \geq 4 \quad \text{for} \quad n = 3 \quad \text{and} \quad d = 1 \,,$$

$$D_1-D_2 \geq 2 \quad \text{or} \quad D_o-D_1 \geq 3 \quad \text{or} \quad D_2-D_3 \geq 3 \quad \text{or}$$

$$D_o-D_1 \,,\; D_2-D_3 \geq 2 \quad \text{for} \quad n = 4 \quad d = 1 \,."$$

"Lemma 3. A third order P - polynomial of degree n has exactly n different real roots, these are: one in (D_o, D_o+1), two in each of (D_{2i}, D_{2i-1}), and one in $(-\infty, D_{n-1})$ if n is even."

"Lemma 4. A third order P - polynomial with the property $D_o-D_i \geq 2d(n-1)$ $i = 1,\ldots,n-1$ is irreducible in the field of rationals."

For each $n \geq 1$, let $d = 1$, $D_o = 0$, $D_1 = -2n$, $D_i = D_{i-1} - 2$ for $i = 2,\ldots,n$. Then $F(x) = x(x-D_1)\ldots(x-D_n) - 1$ is a third order P - polynomial satisfying the hypothesis of lemma 4. Thus $F(x)$ is a monic polynomial of degree $n+1$ with integer coefficients, irreducible over Q and with

constant term -1 . Moreover it has $n+1$ different real roots: one in the interval $(0,1)$, two in each of the intervals (D_{2i},D_{2i-1}) and one in the interval $(-\infty,D_n)$ if $n+1$ is even, i.e. it has one positive real root and n distinct negative real roots. The polynomial $f(x) = (-1)^{n+1}F(-x)$ has all the properties we require: it is monic of degree $n+1$ with integer coefficients, irreducible over Q , with constant term equal to ± 1 , and has exactly n different positive real roots.

CHAPTER V

EXTENSIONS OF PARTIAL ORDERS

5.1 O^* - Groups. The question whether some partial orders of a group can be extended to a partial order that satisfies a given condition can give rise to the definition of several new classes of groups. The one which has received the most attention is the class of O^* - groups.

Definition. An O^* - group is a group in which every partial order can be extended to a full order, or, equivalently, a group in which every maximal partial order is linear.

An alternative definition is offered by the following characterization due to Ohnishi [32].

5.1.1 A group G belongs to the class O^* if and only if it satisfies the following two conditions:

(a) $G \in R^*$ and

(b) for all $g \in G$ and $x,y \in S(g)$, $S(x) \in S(y) \neq \phi$.

In [17] Hollister pointed out that condition (b) is equivalent to

(b') for all $g,x,y \in G$, if x is the product of n conjugates of g and y the product of m conjugates of g , then for some integer k some product of km conjugates of x is equal to some product of kn conjugates of y .

5.2 Closure Properties of the Class 0^* .

5.2.1 $0^* = \{L,D\}0^*$.

The L - closure follows from 5.1.1, a proof of the D - closure can be found in [10].

5.2.2 (Mal'cev [26]). $LN \cap F^{-S} \subseteq 0^*$.

5.2.3 (Kargapolov [19], Fuchs, Sasiada [11]). Non-abelian free groups are not 0^* - groups.

5.2.4 The class 0^* is not closed under any of the closure operations S,P,Q,C,W,F,R .

The S - closure and the W - closure were disproved by Kopytov [23]. Since property (b) of 5.1.1 is preserved under homomorphisms, a quotient of an 0^* - group is an 0^* - group if and only if it belongs to R^* . Gupta and Rhemtulla showed in [12] that not even central extensions of 0^* - groups by 0^* - groups need be 0^* . The example given is the free centre-by-metabelian group on two generators. The C - closure was disproved by Kargapolov in [19]; the proof is reported in [10]. The lack of F - closure and R - closure follows from 5.2.2 and 5.2.3.

Theorem 5.2.5. 0^* is not R_o - closed.

Proof. Consider the group

$$A = \langle a_i \ , \ i \in Z \ ; \ [a_i,a_j] = c_{j-i} \ , \ c_k \in Z(A) \rangle \ .$$

A is nilpotent of class 2 and has an automorphism t defined by $t : a_i \to a_{i+1}$. Let

$$G = \langle A, t \; ; \; a_i^t = a_{i+1} \rangle \quad .$$

Note that the centre of G is equal to $C = \langle c_k \; ; \; k = 1,2,\ldots \rangle$. We will show that $G \in R_o 0^* \setminus 0^*$.

In order to show that $G \notin 0^*$ consider the elements $g = a_1 a_o = a_o^t a_o$ and $g\, c_1 = a_o a_1 = a_o a_o^t$. Both g and gc_1 are products of two conjugates of a_o , hence if $G \in 0^*$, by condition (b') of 5.1.1

$$g^{x_1 t^{\alpha_1}} \cdots g^{x_r t^{\alpha_r}} = g^{y_1 t^{\beta_1}} \cdots g^{y_r t^{\beta_r}} c_1^r \qquad (1)$$

for some $x_i, y_i \in A$, $\alpha_i, \beta_i \in Z$, $r \geq 1$. Without loss of generality, we can assume $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_r$. From (1) it follows that

$$g^{t^{\alpha_1}} \cdots g^{t^{\alpha_r}} \equiv g^{t^{\beta_1}} \cdots g^{t^{\beta_r}} \pmod{C} \quad ,$$

$$a_{\alpha_1+1} a_{\alpha_1} \cdots a_{\alpha_r+1} a_{\alpha_r} \equiv a_{\beta_1+1} a_{\beta_1} \cdots a_{\beta_r+1} a_{\beta_r} \pmod{C} \quad .$$

From this we deduce that $\alpha_i = \beta_i$ and therefore

$$(g^{-u_1} g^{v_1})^{t^{\alpha_1}} \cdots (g^{-u_r} g^{v_r})^{t^{\alpha_r}} = c_1^r \quad , \qquad (2)$$

where u_i and v_i are suitable elements of A . Note that since $C = Z(G)$,

$$(g^{-u_i}g^{v_i})^{t^{\alpha_i}} = (g^{-1}g^{v_i u_i^{-1}})^{u_i t^{\alpha_i}} = [g,v_i u_i^{-1}]^{u_i t^{\alpha_i}} = [g,v_i u_i^{-1}] \quad .$$

Therefore (2) becomes

$$[g,a_{\lambda_1}^{\rho_1} \cdots a_{\lambda_m}^{\rho_m}] = c_1^r \quad , \tag{3}$$

for suitable indices $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ and exponents $\rho_i \neq 0$, $m > 0$ because $c_1^r \neq e$. Since $[g,a_{-i}] = [g,a_{i+1}^{-1}]$ for all i , we can also assume $\lambda_1 \geq 1$. Expanding (3) we obtain

$$c_{\lambda_1-1}^{\rho_1} c_{\lambda_1}^{\rho_1} \cdots c_{\lambda_m-1}^{\rho_m} c_{\lambda_m}^{\rho_m} = c_1^{r_1} \quad ,$$

which is not possible.

In order to show that $G \in R_o O^*$, consider the subgroups

$$C_1 = \langle c_k \ , \ 2^{(2^{2i})} \leq k < 2^{(2^{2i+1})} \quad , \quad i = 0,1,2,\ldots\rangle$$

and

$$C_2 = \langle c_1 \ , \ c_k \ , \ 2^{(2^{2i+1})} \leq k < 2^{(2^{2i+2})} \ , \ i = 0,1,2,\ldots\rangle \quad .$$

Clearly C_1 and C_2 are disjoint normal subgroups of G . We shall prove that G/C_1 and $G/C_2 \in O^*$. By symmetry it is enough to study the group G/C_1 , and to simplify the notation let us call it G . Since $C \in F^{-s} \cap A$ and G/C is the wreath product of two infinite cyclic groups, $G \in O$. Since $G/A \in A$, in order to show that $G \in O^*$ it is sufficient to show that for all $a \in A$ and $x,y \in S(a)$, $S(x) \cap S(y) \neq \phi$. In fact this will assure that every maximal partial order of G is linear on G' and there-

fore linear on G (see [10] p. 95). Let $x = a^{g_1 t^{\beta_1}} \cdots a^{g_r t^{\beta_r}}$, $y = a^{h_1 t^{\gamma_1}} \cdots a^{h_\ell t^{\gamma_\ell}}$, where $g_1,\ldots,g_r,h_1,\ldots,h_\ell \in A$ and $\beta_1,\ldots,\beta_r,\gamma_1,\ldots,\gamma_\ell \in Z$. Then $x \equiv a^{t^{\beta_1}} \cdots a^{t^{\beta_r}} \pmod{C}$ and $y \equiv a^{t^{\gamma_1}} \cdots a^{t^{\gamma_\ell}} \pmod{C}$. By replacing if necessary x with $x^{t^{\gamma_1}} \cdots x^{t^{\gamma_\ell}}$ and y with $y^{t^{\beta_1}} \cdots y^{t^{\beta_r}}$, we can assume $x \equiv y \pmod{C}$. Thus $x = a_{i_1}^{\alpha_1} \cdots a_{i_s}^{\alpha_s} c$ and $y = x\, c_{j_1}^{\delta_1} \cdots c_{j_m}^{\delta_m}$, where $i_1 < \cdots < i_s$, $c \in C$ and $\alpha_1,\ldots,\alpha_s,\delta_1,\ldots,\delta_m \in Z$. The rest of the proof depends on the following

<u>Lemma 5.2.6</u>. In the same context as above, if $x = a_{i_1}^{\alpha_1} \cdots a_{i_s}^{\alpha_s} c$ and $y = x\, c_j^{\delta}$, then $S(x) \cap S(y) \neq \phi$.

Assume lemma 5.2.6 and assume also, by induction, that $x = a_{i_1}^{\alpha_1} \cdots a_{i_s}^{\alpha_s} c$ and $y = x\, c_{j_1}^{\delta_1} \cdots c_{j_{m-1}}^{\delta_{m-1}}$ imply $S(x) \cap S(y) \neq \phi$. Then if $x = a_{i_1}^{\alpha_1} \cdots a_{i_s}^{\alpha_s} c$ and $y = x\, c_{j_1}^{\delta_1} \cdots c_{j_m}^{\delta_m}$ we have by induction that $S(x) \cap S(y\, c_{j_m}^{-\delta_m}) \neq \phi$. Thus $x^{g_1} \cdots x^{g_u} = (y\, c_{j_m}^{-\delta_m})^{h_1} \cdots (y\, c_{j_m}^{-\delta_m})^{h_v} = y^{h_1} \cdots y^{h_v} c_{j_m}^{-\delta_m v}$, for some $g_1,\ldots,g_u,h_1,\ldots,h_v \in G$. Let $x' = x^{g_1} \ldots x^{g_u}$ and $y' = y^{h_1} \cdots y^{h_v}$, then $y' = x'\, c_{j_m}^{\delta_m v}$ and by lemma 5.2.6 $S(x') \cap S(y') \neq \phi$. Since $S(x') \subseteq S(x)$ and $S(y') \subseteq S(y)$, we conclude

that $S(x) \cap S(y) \neq \phi$.

<u>Proof of Lemma 5.2.6</u>. Without loss of generality we can assume $c = e$, for suppose that $\bar{x} = a_{i_1}^{\alpha_1} \cdots \alpha_{i_s}^{\alpha_s}$ and $\bar{y} = \bar{x}\, c_j^{\delta_j}$ imply $\bar{x}^{g_1} \cdots \bar{x}^{g_r} = \bar{y}^{h_1} \cdots \bar{y}^{h_q}$ for some $g_1,\ldots,g_r,h_1,\ldots,h_q \in G$. Then by considering the last identity modulo C we can see that $r = q$ and therefore $x^{g_1} \cdots x^{g_r} = \bar{x}^{g_1} \cdots \bar{x}^{g_r} c^r = \bar{y}^{h_1} \cdots \bar{y}^{h_r} c^r = y^{h_1} \cdots y^{h_r}$. Thus we have $x = a_{i_1}^{\alpha_1} \cdots a_{i_s}^{\alpha_s}$, $y = x\, c_j^{\delta_j}$, $i_1 < \cdots < i_s$. By conjugating if necessary by a suitable power of t we can assume that $i_1 = 0$. Write $i = i_s$. Define

$$x_1 = x^{a_{i+j}^{\sigma_1}} = x \prod_{k=1}^{s} c_{i+j-i_k}^{\sigma_1 \alpha_k} ,$$

where σ_1 is an integer to be chosen later. Let j_2 and $j^{(2)}$ be respectively the minimum and the maximum of the subscripts of the c's occurring in the expression of $x^{-1}x_1$ modulo $\langle c_j \rangle$. Note that

$$j+1 \leq j_2 \leq j^{(2)} = i+j .$$

Iterate the procedure by defining

$$x_n = x_{n-1}^{a_{i+j_n}^{\sigma_n}} = x \prod_{k=1}^{s} c_{i+j-i_k}^{\sigma_1 \alpha_k} \cdots \prod_{k=1}^{s} c_{i+j_n-i_k}^{\sigma_n \alpha_k} ,$$

and letting j_{n+1} and $j^{(n+1)}$ be respectively the minimum and the maximum of the subscripts of the c's occurring in the expression of $x^{-1}x_n$ modulo

$<c_j, c_{j_2}, \ldots, c_{j_n}>$. An easy induction shows that

$$j+n \leq j_{n+1} \leq j^{(n+1)} \leq ni + j \ .$$

Take $n = 2^{(2^{2\ell})}$ with ℓ large enough to verify $j < 2^{(2^{2\ell})}$ and $i+1 < 2^{(2^{2\ell})}$ so that we have

$$2^{(2^{2\ell})} \leq j_{n+1} \leq j^{(n+1)} < 2^{(2^{2\ell+1})} \quad ,$$

and therefore $x^{-1}x_n \in <c_j, c_{j_2}, \ldots, c_{j_n}>$.

Consider the homogeneous linear system of $n-1$ equations in the n unknowns $\sigma_1, \ldots, \sigma_n$ consisting of the conditions that the exponents of $c_{j_2}, \ldots, c_{j_n}$ in the expression of $x^{-1}x_n$ be all zero. Observe that the system is of the form

$$\begin{pmatrix} * & * & 0 & \cdots & 0 \\ * & * & * & \cdots & 0 \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ * & * & * & \cdots & * \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \cdot \\ \cdot \\ \sigma_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

hence in any non-zero solution $\sigma_1 \neq 0$. Choose a non-zero integer solution $\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n$ such that $\bar{\sigma}_1 \alpha_s$ has the same sign as δ and $\delta | \bar{\sigma}_1 \alpha_s$. Let $\bar{\sigma}_1 \alpha_s = \delta m$. Then

$$x^{m-1} x^{a_{i+j}^{\bar{\sigma}_1} a_{i+j_2}^{\bar{\sigma}_2} \cdots a_{i+j_n}^{\bar{\sigma}_n}} = x^m x^{-1} x_n = x^m c_j^{\delta m} = (x\, c_j^{\delta})^m = y^m \quad ,$$

which shows that $S(x) \cap S(y) \neq \phi$.

5.3 Which Classes of Groups are Contained in 0^* ? The main open question regarding 0^* - groups is under what conditions does an 0 - group or an R^* - group belong to 0^* . We have the following partial answers.

5.3.1 [12]. Every metabelian R^* - group is an 0^* - group.

Theorem 5.3.2. Every polycyclic 0 - group is an 0^* - group.

The proof of 5.3.2 is based on the following two lemmas.

Lemma 5.3.3. Let $G \in P\,G_1 \cap 0$ and let $C \prec D$ be a minimal convex jump, in the sense that if K is a relatively convex subgroup of G and $C \leq K \leq D$, then either $C = K$ or $K = D$. If $H \Delta G$ and $C \leq H \leq D$, then either $C = H$ or D/H is finite.

Proof. Since $G \in P\,G_1$, all relatively convex subgroups are normal and we can assume without loss of generality that $C = \{e\}$. Let $\Gamma = C_G(D)$, then $\Gamma \geq G'$. Let $G = \langle \Gamma, \lambda_1, \ldots, \lambda_k \rangle$ and identify D with a subgroup of the additive group of the reals on which λ_i acts as multiplication by the positive algebraic number α_i , for $i = 1, \ldots, k$. Let us show that the isolator I of H is actually strongly isolated. Suppose that

$d \sum_i n_i \alpha_1^{r_{1i}} \cdots \alpha_k^{r_{ki}} \in I$ for some $d \in D$, positive integers n_i and integers r_{ji} . Since $\sum_i n_i \alpha_1^{r_{1i}} \cdots \alpha_k^{r_{ki}} \neq 0$ and the α_i's are algebraic, its inverse has the form $\sum_j q_j \alpha_1^{t_{1j}} \cdots \alpha_k^{t_{kj}}$, where $q_j \in Q$. If m is the l.c.m. of the denominators of the q_j's , then $md = x \sum_j m_j \alpha_1^{t_{1j}} \cdots \alpha_k^{t_{kj}}$, where the m_j's are integers, and $x \in I$. Thus $md \in I$ and therefore $d \in I$.

Since I is a strongly isolated normal subgroup of G contained in D which is normal, relatively convex and G - fully orderable, by 1.1.6 I is relatively convex in G. If $H \neq C$, then $I \neq C$, hence hy hypothesis $I = D$ and therefore $D/H \in \mathcal{F}$.

<u>Lemma 5.3.4</u>. Let $G \in P\mathcal{G}_1 \cap \mathcal{O}$, $F = \mathrm{Fitt}(G)$, $E \leq Z(F)$, E isolated and normal in G. Then $G/E \in \mathcal{O}$.

<u>Proof</u>. Clearly it is sufficient to prove the assertion in the case when E is a minimal isolated and normal subgroup of G contained in $Z(F)$. Let

$$\{e\} \prec C_1 \prec C_2 \prec \cdots \prec F \prec \cdots \prec G$$

be a chief convex series, i.e. a convex series in which all the jumps are minimal in the sense of lemma 5.3.3, and let r be the first integer such that $E \cap C_r \neq \{e\}$. Write C for C_{r-1} and D for C_r. Since $D \cap E$ is normal and isolated, $D \cap E = E$, i.e. $E \leq D$. Suppose that $G/E \notin \mathcal{O}$, since $G/D \in \mathcal{O}$ this implies that D/E is not G/E - orderable, therefore by 1.2.3 there exist $g_1,\ldots,g_r \in D/E$ such that $S(g_1^{\varepsilon_1},\ldots,g_r^{\varepsilon_r}) \cap E \neq \phi$ for all choices of signs $\varepsilon_i = \pm 1$. By lemma 5.3.3, $D/EC \in \mathcal{F}$, thus there exists an integer $n > 0$ such that $g_i^n = x_i y_i$, with $x_i \in E$, $y_i \in C$. Since E is isolated $y_i \neq e$ and without loss of generality we can assume $y_i > e$ for $i = 1,\ldots,r$. Let $f \in S(g_1,\ldots,g_r) \cap E$, then $f = g_1^{h_{11}} \cdots g_1^{h_{1i_1}} \cdots g_s^{h_{s1}} \cdots g_s^{h_{si_s}}$. $f^n = g_1^{nh_{11}} \cdots g_s^{nh_{si_s}} d$, for some $d \in J'$, where $J = \langle g_1,\ldots,g_s\rangle^G$. Since $[E,C] = \{e\}$, $f^n = x_1^{h_{11}} \cdots x_s^{h_{si_s}} y_1^{h_{11}} \cdots$

$y_s^{h_{si_s}} d = xyd$, where $x = x_1^{h_{11}} \cdots x_s^{h_{si_s}} \in E$ and $y = y_1^{h_{11}} \cdots y_s^{h_{si_s}} \in C$. Let j be the least integer such that $\langle y_1,\ldots,y_s\rangle \leq C_j$. Note that since $y \geq y_1^{h_{11}},\ldots,y_s^{h_{si_s}}$, $y \in C_j \setminus C_{j-1}$. We now show that $d \in C_{j-1}$. $[g_i^n,J] = [x_iy_i,J] = [y_i,J] \subseteq C_{j-1}$ and since $G/C_{j-1} \in 0$, by 1.6.1, $[g_i,J] \subseteq C_{j-1}$. Since J and C_{j-1} are normal in G we conclude that $J' \leq C_{j-1}$ and $d \in C_{j-1}$. Thus the element $x^{-1}f^n = yd$ is different from e and belongs to $E \cap C$, a contradiction.

<u>Proof of Theorem 5.3.2</u>. Let $G \in P\,\mathcal{G}_1 \cap 0$, $F = \mathrm{Fitt}(G)$ and P a maximal partial order on G , then P is linear on $Z(F)$. Let $\{e\} \prec E \prec \cdots \prec Z(F)$ be the chain of convex subgroups of the order $P \cap Z(F)$. Since E is normal and isolated in G , by 5.3.4 $G/E \in 0$. We show that $\overline{P} = PE/E$ is a partial order on $\overline{G} = G/E$. Clearly it is a normal semigroup, we must check that $\overline{P} \cap \overline{P}^{-1} = \{\overline{e}\}$. Suppose that $p_1^{-1}x = p_2$, where $p_1,p_2 \in P$, $x \in E$, then $x \geq p_1 \geq e$. Write p for p_1 . If $p \in F$, then by extending P to a full order Q on F we see that $p \in E$. We show that p must indeed belong to F . Since $\langle E,p\rangle \in \mathcal{A}^2 \cap 0$, by 5.3.1 $P \cap \langle E,p\rangle$ can be extended to a full order. In such an order either $p \sim x$ or $p \ll x$, in both cases $|[p,x]| \ll x$ and since $[p,x] \in E$ which is archimedeanly ordered $[p,x] = e$. Thus in both cases $[p,E] = \{e\}$ and hence $[\langle p\rangle^{\langle G\rangle},E] = e$. Let $a \in F$, either $[p,a] \in P$, or $[p,a] \in P^{-1}$, or $[p,a] \notin P \cup P^{-1}$. If $[p,a] \in P$, from $x \geq p \geq e$ we obtain $x^a = x \geq p^a = p[p,a] \geq [p,a] \geq e$ which implies $[p,a] \in E$. If $[p,a] = a^{-1p}a \in P^{-1}$ note that $[p,a^{-1}] = a^pa^{-1} = (a^{-1}a^p)^{a^{-1}} \in P$. The previous argument applied to $[p,a^{-1}]$ gives $[p,a^{-1}] \in E$, hence $[p,a] \in E$ also. If $[p,a] \notin P \cup P^{-1}$,

then $[p,a]^{g_1} \cdots [p,a]^{g_n} \in P$ for some $g_i \in G$. From $x \geq p$ we obtain $x \geq [p,a]$ and hence $x^{g_1} \cdots x^{g_n} \geq [p,a]^{g_1} \cdots [p,a]^{g_n} \geq e$ which implies $[p,a]^{g_1} \cdots [p,a]^{g_n} \in E$. Since E is strongly isolated we conclude that $[p,a] \in E$. Thus $[p,F] \subseteq E$ and hence $[\langle p\rangle,F] \leq E$. Also $[\langle p\rangle,E] = \{e\}$, as shown above. This implies that $\langle p,F\rangle$ is a normal nilpotent subgroup of G , but since $F = \mathrm{Fitt}(G)$, it must be $p \in F$.

By induction on the policyclic length of G , $\overline{P}$ can be extended to a full order $\overline{Q}$ of $\overline{G}$. $P \cap E$ and $\overline{Q}$ define a linear order on G which is an extension of P . Since P is maximal, P itself is linear.

5.4 I^* - Groups. This class of groups has been introduced in [9] by Fuchs who asked: in which groups can every partial order to extended to an isolated partial order?

Definition. An I^* - group is a group in which every partial order can be extended to an isolated partial order, or, equivalently, a group in which every maximal partial order is isolated.

5.4.1 (Hollister [16]). A group G belongs to the class I^* if and only if it satisfies the following two conditions:

(a) G is torsion-free and

(b) for all $x \in G$ if $t \in S(x)$ and $n > 0$, then $e \in S(t^{-1},x^n)$.

If G is an R^* - group then (b) is equivalent to

(b') for all $x \in G$, if $t \in S(x)$ and $n \in 0$, then $S(t) \cap S(x^n) \neq \phi$.

While 0^* is contained in I^*, I^* neither contains nor is contained in 0, for free groups on more than one generator belong to $0 \backslash I^*$ and torsion-free groups with only two conjugacy classes belong to $I^* \setminus 0$ [16]. It is not yet known whether 0^* is contained in $I^* \cap 0$ properly.

5.5 Closure Properties of the Class I^*.

5.5.1 The class I^* is L - closed and is not closed under any of the operations $0, S, P, F, R$.

Proof. From 5.4.1 it follows that I^* is L - closed and that a quotient of an I^* - group is I^* if and only if it is torsion-free. The lack of S - closure follows from the fact that every torsion-free group can be embedded in a torsion free group with just two conjugacy classes. Theorem 5.6.3 shows that central extensions of I^* - groups by I^* - groups need not be I^*. Since $F^{-s} \cap N$ - groups are I^*, while free groups on more than one generator are not, I^* is neither F - closed nor R - closed.

It is not yet known whether I^* is closed under direct and wreath products. The next theorem is an improvement on the result that I^* is not S - closed.

Theorem 5.5.2. A subgroup of an 0^* - group need not be I^*.

Proof. We give an example built on the pattern of Kopytov's in [23]. Let

$$A_o = \langle a_i, c, i\in Z; [a_i,c] = e\ ,\ [a_i,a_j] = \begin{cases} e & \text{if } j-i \equiv 0 \pmod 3 \\ c & \text{if } j-i \equiv 1 \pmod 3 \\ c^{-1} & \text{if } j-i \equiv 2 \pmod 3 \end{cases} \rangle .$$

A_o has an automorphism t defined by $a_i^t = a_{i+1}$, $c^t = c$. Let

$$G_o = \langle A_o, t\ ;\ a_i^t = a_{i+1}\ ,\ c^t = c\rangle \ .$$

It is shown in the proof of 5.6.3 that the group

$$A = \langle a,b,t\ ;\ a^t = b\quad b^t = (ba)^{-1}\ ,\ [a,b] = c \in Z(A)\rangle$$

is torsion-free but not I^* . Since A is a homomorphic image of G_o under the mapping ϕ defined by $a_o\phi = a$ and $t\phi = t$, G_o cannot be an I^* - group either. Now embed G_o in the group

$$G = \langle G_o, u_i, i\in Z\ ;\ [u_i,u_j] = [u_i,c] = e\ ,\ u_i^t = u_{i+1}, [a_i,u_j] = \begin{cases} e & \text{if } i \neq j \\ c & \text{if } i = j \end{cases} \rangle .$$

We show that G is an 0^* - group. Note that $G \in 0$, for $Z(G) = \langle c\rangle$ and $G/Z(G)$ is isomorphic to the wreath product of a free abelian group of rank 2 and an infinite cyclic group. Let $H = \langle A_o, u_i, i\in Z\rangle$. Since $G' \leq H$, in order to show that $G \in 0^*$ it is sufficient to show that for all $x \in H$ and $x_1, x_2 \in S(x)$, $S(x_1) \cap S(x_2) \neq \phi$. If $x \in \langle c\rangle$ this is obvious. Let $x \in H\backslash\langle c\rangle$. Since $G/\langle c\rangle \in A^2 \cap 0$, $G/\langle c\rangle \in 0^*$ by 5.3.1, hence there exists $y_1 \in S(x_1)$ and $y_2 \in S(x_2)$ such that $y_2 = y_1 c^n$ for some integer n . If $n = 0$ the assertion that $S(x_1) \cap S(x_2) \neq \phi$ is proved. Suppose that $n \neq 0$ Since $y_1 \in S(x)$, $y_1 \notin \langle c\rangle$ and there exists $h \in H$ such that $y_1^h = y_1 c^m$

with $m > 0$. Then $y_1^{m-1} y_1^{h^n} = y_1^m c^{mn} = y_2^m$, i.e. $S(y_1) \cap S(y_2) \neq \phi$ and consequently $S(x_1) \cap S(x_2) \neq \phi$.

5.6 Which Classes of Groups are Contained in I^* ?

Trying to answer Fuchs' original question we prove the following.

5.6.1 Polycyclic 0 - groups are I^* .

Proof. This is just a corollary of 5.3.2.

Theorem 5.6.2. Torsion-free abelian-by-nilpotent groups are I^* .

Theorem 5.6.3. Torsion-free polycyclic groups, centre-by-metabelian 0 - groups and abelian-by-polycyclic 0 - groups need not be I^* .

The proof of 5.6.2 depends on the following.

Lemma 5.6.4. Let P be a maximal partial order on a group G and H a normal subgroup of G . If $H \in I^*$ then $P \cap H$ is an isolated partial order on H .

Proof. Let A be the intersection of all maximal partial orders on H extending $P \cap H$. Then A is isolated; moreover it is normal in G so that AP is a normal subsemigroup of G . In fact AP is a partial order of G , for if $a_1 p_1 = a_2^{-1} p_2^{-1}$, with $a_1, a_2 \in A$ and $p_1, p_2 \in P$, then $a_2 a_1 = p_2^{-1} p_1^{-1} \in A \cap P^{-1} \subseteq A \cap A^{-1} = \{e\}$. The maximality of P implies that $A \subseteq P$ and therefore $P \cap H = A$ is isolated.

Proof of Theorem 5.6.2. By 5.3.1, free metabelian groups are I^*, hence every torsion-free metabelian group is I^*. Assume by way of induction that $F^{-s} \cap AN_{c-1} \subseteq I^*$, and let $G \in F^{-s} \cap AN_c$. Suppose that P is a maximal partial order on G and that $g^n \in P$ for some $g \in G$ and $n > 0$. The group $H = \langle G', g \rangle$ is normal in G and belongs to $F^{-s} \cap AN_{c-1} \subseteq I^*$. By lemma 5.6.4, $P \cap H$ is isolated hence $g \in P \cap H \subseteq P$.

Proof of Theorem 5.6.3. First we show that the group

$$G = \langle a,b,t \ ; \ a^t = b \ , \ b^t = (ba)^{-1} \ , \ t^9 = [b,a] \rangle$$

is not I^*. Let $P = S(t^3)$. Since $t^3 \in Z(G)$, P is a partial order on G, and it cannot be extended to an isolated partial order because $t^5 t^{b^{-1}} t^2 t^b = e$. Clearly G is polycyclic, and it was shown in [3] that it is torsion-free.

Since G is torsion-free centre-by-metabelian on two generators, we see that the free centre-by-metabelian group on two generators, which is ordered by 1.5.2 does not belong to I^*.

Finally let

$$A = \langle a,b,t \ ; \ a^t = b \ , \ b^t = (ba)^{-1} \ , \ [[a,b],a] = [[a,b],b] = e \rangle \ ,$$

and let F/K be a representation of A as a quotient of a free group. F/K' is abelian-by-polycyclic and by 6.2.5 it is ordered. Since F/K' has a quotient isomorphic to G, $F/K' \notin I^*$.

CHAPTER VI

RIGHT-ORDERED GROUPS

6.1 Introduction. A group G is called right-ordered (or a RO - group) if it is possible to define in G an order relation $\leq$ stable under right multiplication, i.e. such that $a \leq b$ implies $ac \leq bc$ for all $a,b,c \in G$. This is equivalent to the existence of a subsemigroup P of G such that $P \cap P^{-1} = \phi$ and $P \cup P^{-1} \cup \{e\} = G$. In fact, given the order relation $\leq$, P can be defined as the set of positive elements and viceversa, given P, we can define $a \leq b$ if and only if $ba^{-1} \in P$. By dropping the condition $P \cup P^{-1} \cup \{e\} = G$, we would obtain a partial right-order.

Every group admitting a right-order admits a left-order as well: let $\leq$ be a right-order in G and P its positive cone, then by defining $a \leq' b$ if and only if $a^{-1}b \in P$ we obtain an order relation stable under left mutliplication. This is why it is customary not to consider left-orders.

Right-ordered groups are always torsion-free but not necessarily R - groups, e.g. $G = \langle a,b \, ; \, a^b = a^{-1} \rangle$ is right-ordered because it is the extension of an infinite cyclic group by an infinite cyclic group, and $P\,RO = RO$ (see 6.4.2), but it is not an R - group because $(ab)^2 = b^2$. An example of a torsion-free group which is not RO is given in [37], more examples can be constructed using theorem 1 in [33].

6.2 Characterizations of RO - Groups. For each subset A of a group G, let (A) denote the semigroup generated by A. Several characterizations

of RO - groups have been given by Conrad in [7]. We only mention the following

6.2.1 A group G is right-ordered if and only if for every finite set $\{x_1,\ldots,x_n\}$ of elements in $G\backslash\{e\}$ there are signs $\varepsilon_i = \pm 1$ such that $(x_1^{\varepsilon_1},\ldots,x_n^{\varepsilon_n}) \not\ni e$.

6.2.2 (Cohn, Conrad, [6][7]). The class of RO - groups coincides with the class of groups of order-preserving permutations of an ordered set.

6.2.3 (Fried, Holland [8], [15]). The class of RO - groups coincides with the class of subgroups of lattice-ordered groups.

6.2.4 (Smirnov [36]). Let $G = F/A$, where F is free. Then G is RO if and only if A/A' is F/A' - orderable.

It is worth while mentioning also the following result connected with the above.

6.2.5 (Smirnov [36]). Let F be free and $A \Delta F$, if F/A has an infrainvariant system with torsion-free abelian factors, then $F/A' \in 0$. If F/A has an infrainvariant solvable system, then $F/A' \in RO$ and $F/A'' \in 0$.

6.3 Convex Subgroups. Just as with two-sided orders, we define a right-order P to be archimedean if for every pair $a,b \in P$ there is a positive integer n such that $a < b^n$. A subgroup C of a RO - group G is convex if for every $g \in G$ and $c \in C$, $e \leq g \leq c$ implies $g \in C$.

6.3.1 (Conrad [7]). An archimedean right-order is a two-sided order, thus an archimedeanly right-ordered group is isomorphic to a subgroup of the additive group of the reals and does not have any proper convex subgroups.

The converse is not true.

6.3.2 (Smirnov [36]). Let G be the multiplicative group of 2×2 rational matrices of the form $\begin{pmatrix} k & a \\ 0 & 1 \end{pmatrix}$ with $k > 0$, and let it be right-ordered with positive cone $P = \{\begin{pmatrix} k & a \\ 0 & 1 \end{pmatrix} \mid k+a\varepsilon \geq 1\}$, where ε is any fixed positive irrational number. Then G has no proper convex subgroup although it is not archimedeanly ordered.

6.4 <u>Closure Properties of the Class of</u> RO - <u>Groups</u>.

6.4.1 If a group G has a normal system with RO factors, then G is RO.

<u>Proof</u>. Let $\sum$ be such a system. Every $g \in G$ determines a jump $C_g \prec C^g$, where $C_g = \langle C \in \sum \mid C \not\ni g \rangle$ and $C^g = \cap\{C \in \sum \mid C \ni g\}$. Define P to be the set of all the $g \in G$ such that $C_g\, g$ is positive in the right-order of C^g/C_g. Then P is a right-order on G.

6.4.2 $RO = \{S,L,P,D,C,F,R,W,\overline{W}\}\ RO$, $RO \neq Q\ RO$.

<u>Proof</u>. The S - closure is obvious, the L - closure follows from the local character of 6.2.1, all the rest are consequences of 6.4.1. A quotient G/H of a RO - group G is RO if and only if H is relatively convex.

6.5 The Classes C and C^* .

6.5.1 (Conrad [7]). Let P be a right order on a group G . Then the following are equivalent:

(1) for all $a,b \in P$, there exists a positive integer n such that $(ab)^n > ba$;

(2) for all $a,b \in P$, if $a < b$ there exists a positive integer n such that $ab^n a^{-1} > b$;

(3) for all $a,b \in P$, there exists a positive integer n such that $a^n b > a$;

(4) if $C \prec D$ is a convex jump in G , then $C \Delta D$ and D/C is isomorphic to a subgroup of the additive group of the reals.

We will denote by C the class of groups admitting a right order that satisfies one of the above conditions. Thus C coincides with the class of groups having a normal system with torsion-free abelian factors and contains the class of O - groups. We list here a few more equivalent formulations of the condition in 6.5.1:

(5) for all $a,b \in P^{-1}$, there exists a positive integer n such that $(ab)^n < ba$;

(6) for all $a,b \in P^{-1}$, if $a > b$ there exists a positive integer n such that $ab^n a^{-1} < b$;

(7) for all $a,b \in P^{-1}$, there exists a positive integer n such that $a^n b < a$;

(8) for all $a,b \in P$, $a << b$ if and only if $b^{-1} << a^{-1}$;

(9) for all $a,b \in P$, $a << b$ implies $b^{-1} < a^{-1}$;

(10) for all $a,b \in P^{-1}$, $b << a$ implies $a^{-1} < b^{-1}$;

(11) for all $a,b \in G$ and $c \in P$, if $|a| << c$ and $|b| << c$, then $|ab| << c$;

(12) for all $c \in P$, the set $\{x \in G \mid |x| << c\}$ is a convex subgroup of G.

Proof. Note that (5), (6), (7) and (10) are respectively the dual of (1), (2), (3) and (9), i.e. they can be deduced from each other by substituting the right-order P with its opposite P^{-1}, and in this sense (4) is self-dual. Thus 6.5.1 implies the equivalence of (1) through (7).

We now show the following implications:

$$(1) + (5) \Rightarrow (8) \Rightarrow (9) \Rightarrow (3) .$$

Let $a,b \in P$, $a << b$. We must show that $b^{-1} << a^{-1}$. If this were not so, then for some positive integer m we would have $a^{-m} < b^{-1}$, and by applying (1) to the elements b and $b^{-1}a^{m}$ we would obtain $a^{mn} > b^{-1}a^{m}b > b$, which contradicts $a << b$. In the same way, using (5) instead of (1), we see that $b^{-1} << a^{-1}$ implies $a << b$. Obviously (8) implies (9). To prove that (9) implies (3), consider two positive elements a and b. If $a \leq b$, property (3) is satisfied with $n = 1$. Let $a > b$ and suppose that for all positive integers n, $a^{n}b < a$. Then $a << ab^{-1}$, and by (9) $ba^{-1} < a^{-1}$ contradicting the fact that b is positive. Since (10) is the dual of (9), we have proved the equivalence of (1) through (10).

In order to show that (1)-(10) imply (11) we need the following result from [7].

6.5.2 Let P be a right-order on a group G satisfying conditions (1)-(3) and let $x \in G$ and $a,y \in P$. If $x < a^m$ and $y < a^n$ for some positive integers m and n, then there exists a positive integer q such that $xy < a^q$.

Let $a,b \in G$ and $c \in P$ and assume that $|a| \ll c$ and $|b| \ll c$. We must show that $|ab| \ll c$. If $b = a^n$ for some $n \in Z$, the assertion is obvious.

<u>Case 1</u>. $a > b > e$. Then $ab^{-1} \in P$ and applying (3) to the elements a and ab^{-1} we obtain that $a^n ab^{-1} > a$, $a^{n+1} > ab$ for some positive integer n. By 6.5.2 since ab is less than some power of a, any power of ab is less than some power of a, hence less than c.

<u>Case 2</u>. $b > a > e$. Then $b^2 > ab$ and again by 6.5.2 any power of ab is less than some power of b, hence less than c.

<u>Case 3</u>. $a < e$, $b < e$. Then $a^{-1} \ll c$ and $b^{-1} \ll c$ and by cases 1 and 2 $b^{-1}a^{-1} = |ab| \ll c$.

<u>Case 4</u>. $a > e > b$, $ab > e$. Applying (1) to ab and b^{-1} we obtain $a^n > b^{-1}ab > ab$ for some positive integer n. Thus any power of ab is less than some power of a, hence less than c.

<u>Case 5</u>. $a > e > b$, $ab < e$. Then $b^{-1} > e > a^{-1}$ and $b^{-1}a^{-1} > e$. By case 4 $b^{-1}a^{-1} = |ab| \ll c$.

Case 6. $b > e > a$, $ab > e$. Then $b > ab$, and any power of ab is less than some power of b , hence less than c .

Case 7. $b > e > a$, $ab < e$. Then $a^{-1} > e > b^{-1}$, $b^{-1}a^{-1} > e$ and by case 6 $b^{-1}a^{-1} = |ab| << c$.

(11) implies (1). Let $a,b \in P$ and suppose that $(ab)^n < ba$ for all positive integers n , i.e. $ab << ba$. Note that $b \not<< ba$, for otherwise $|b^{-1}| << ba$ and $|ab| << ba$ would imply $|a| = |abb^{-1}| << ba$ and hence $ba << ba$. Let n be the least positive integer such that $ba < b^n$. From $e < a$ we obtain $b < ab$ and hence $bab < (ab)^2 < ba < b^n$ and $ba < b^{n-1}$, contradicting our choice of n .

(1)-(11) imply (12). Let $c \in P$, by (11) the set $C = \{x \mid |x| << c\}$ is a subgroup of G . Suppose that $e < y < x << c$. By 6.5.2 any power of y is less than some power of x , hence less than c , i.e. $y \in C$. Conversely it is clear that (12) implies (11).

6.5.3 $C = \{S,L,P,D,C,W,\overline{W},F,R\} C$.

Proof. The S - closure is obvious, the L - closure can be proved by imitating Kurosh's proof of the L - closure of SN - groups in [24], p. 183, the rest follow from the fact that a group having a normal system with C - factors is itself a C - group.

Let us denote by C^* the class of RO - groups all of whose right-orders are C - right-orders.

6.5.4 (Ault, Rhemtulla [1],[33]) $(F^{-s} \cap LN)$ - groups are C^* - groups.

From the proof in [33] it can be seen that $RO \cap (LN)P \subseteq C^*$, and since $C^* = LC^*$, also $RO \cap L(NP) \subseteq C^*$.

6.5.5 Metanilpotent RO - groups are C - groups.

Proof. Let $G \in RO$, $H \Delta G$, H and $G/H \in N$. Let T/H be the torsion subgroup of G/H, then $T \in RO \cap NP$ and $G/T \in F^{-s} \cap N$ hence $G \in PC^* \subseteq PC = C$.

6.5.6 (Smirnov [36], p. 52). C^* - groups need not be ordered, e.g. $G = \langle a,b \; ; \; a^b = b^{-1} \rangle$.

Theorem 6.5.7. Polycyclic metabelian 0^* - groups need not be C^*.

Proof. Let σ and τ be the following order-preserving transformations of the real line:

$$x\sigma = x + 1 \quad \text{and}$$

$$x\tau = x/\alpha \quad ,$$

where $\alpha = +\sqrt{\dfrac{5+\sqrt{21}}{2}}$ is a root of the equation $x^4 - 5x^2 + 1 = 0$. The group G generated by σ and τ admits the following presentation:

$$G = \langle \sigma,\tau ; \sigma^{\tau^4} = (\sigma^5)^{\tau^2} \sigma^{-1} \, , \, [\sigma,\sigma^{\tau^i}] = e \text{ for } i = 1,2,3 \rangle \quad .$$

G is the extension of $\langle\sigma\rangle^G$, which is the direct product of four infinite cyclic groups, by an infinite cyclic group, thus it is metabelian and polycyclic. If we identify $\langle\sigma\rangle^G$ with the subgroup of the additive group of

the reals generated by $1,\alpha,\alpha^2$ and α^3, we see that τ acts on $\langle\sigma\rangle^G$ as multiplication by α, therefore G is ordered and by 5.3.1 G is an 0^* - group.

In order to show that G is not a C^* - group, we will right-order G in such a way that property (3) of 6.5.1 will not be satisfied. Well-order the set R of real numbers letting 0 be the first element and -1 the second, then for any $\gamma \in G$ consider the first $r \in R$ in the given well-ordering such that $r\gamma \neq r$, and define γ to be positive if $r\gamma > r$ in the natural order of R. In this way, σ,τ and $\sigma\tau$ turn out to be psotive, but $(\sigma\tau)^n \tau(\sigma\tau)^{-1}$ is negative for all positive integers n, because under $(\sigma\tau)^n \sigma^{-1}$ the element 0 is mapped to

$$\left(\frac{1}{\alpha^n} + \frac{1}{\alpha^{n-1}} + \ldots + \frac{1}{\alpha}\right) - 1 < \left(\sum_{i=0}^{\infty} \frac{1}{\alpha^i}\right) - 2 = \frac{2-\alpha}{\alpha-1} < 0 \quad .$$

Thus the right-order that we have imposed on G does not satisfy property (3).

The following are instances in which C - groups behave differently from 0 - groups.

6.5.8 The group

$$G = \langle a,b,t \; ; \; a^b = a^{-1} \; , \; [a,t^2] = [b,t^2] = [a,a^t] = [b,b^t] = [a,b^t] = e\rangle$$

is an example of polycyclic C - group which is not nilpotent-by-abelian.

G admits the normal chain

$$\langle e\rangle \,\Delta\, \langle a\rangle \,\Delta\, \langle a,b\rangle \,\Delta\, \langle a,b,a^t\rangle \,\Delta\, \langle a,b,a^t,b^t\rangle \,\Delta\, G$$

all of whose factors are infinite cyclic, thus G is a polycyclic C-group. G' is not nilpotent because it contains the two elements $x = a^{-1}a^t$ and $y = b^{-1}b^t$ which are subject to the relation $x^y = x^{-1}$.

6.5.9 The group

$$G = \langle x,y,z \;;\; x^2y^{-1}x^2y = y^2x^{-1}y^2x = z \;,\; [x,z] = [y,z] = e\rangle$$

is an example of C- group whose central quotient is torsion-free, but cannot be right-ordered ([33], Theorem 1).

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